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Doctoral Colloquium—A Comparison of Teachers and Students in Categorizing and Representing Geometric Conjectures Using AR Simulations

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Abstract. This study investigates how an extended reality (XR)-based learning environment, GeoGebra MR, supports progressive formalization (PF) in geometric reasoning by comparing the reasoning processes of high school teachers and students. This doctoral colloquium paper explores how participants categorize and represent geometric conjectures through XR interaction. Findings reveal that teachers categorize conjectures by emphasizing underlying principles and concepts, effectively constructing and connecting scientific and algebraic representations with their sensorimotor experiences using GeoGebra MR's interactive features. In contrast, students tend to categorize conjectures based on surface-level details, often remaining at literal and naïve representations or prematurely applying incorrect formal reasoning, struggling to integrate abstract concepts with hands-on experiences. These differences highlight the challenges students face in engaging with PF and underscore the need for scaffolding strategies, such as embedding explicit operational cues in the literal elements of problems and visualizing algebraic features in XR environments. The results provide evidence-based design insights for XR-based learning tools to better support learners' progression from concrete experiences to formal representations, ultimately enhancing geometric reasoning and educational outcomes.

Keywords: Extended Reality (XR), Progressive Formalization (PF), Geometric Reasoning, Categorization, Representation, Four Stages of Representation, GeoGebra MR.

1 Theoretical Framework

1.1 Extended Reality in Geometry Education

Extended reality (XR) simulations, such as *The Hidden Village* (THV) [1, 2] and *GeoGebra MR* [3, 4], exemplify tools that provide concrete, experiential learning opportunities. THV is a virtual reality-based motion-capture simulation designed for embodied geometry education. The system prompts students to perform physical actions connected to geometric conjectures while simultaneously verbalizing their reasoning [1]. Research has shown that THV effectively fosters mathematical insights that facilitate the development of advanced geometric reasoning [2]. GeoGebra MR, the tool used in this study, features both augmented reality and virtual reality environments [5]. As a network-based simulation, it allows users to collaboratively interact with dynamic 3D geometric objects in real time. The HoloLens 2 headset enables users to anchor the virtual objects to a specific common location and to immediately update any physical changes to object location, shape, or orientation to all engaged users. Utilizing hand-tracking functionality, users can observe and manipulate 3D holographic objects (e.g., cubes, cylinders, triangles) with their hands while operating algebraic functions through a floating virtual dashboard (further described in Section 3.2, Materials). Both THV and GeoGebra MR can bridge the gap between concrete

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and algebraic representations by facilitating grounded symbolization through sensorimotor experiences. Research has shown that successful mathematical problem-solving often involves physically interacting with mathematical objects—touching, shifting, and regrouping [6]. These sensorimotor experiences, when combined with transformational speech, serve as developmental precursors to generalized mathematical procedures that can later be internalized and applied mentally [2]. Interactive XR simulations are particularly effective because they encourage learners to generate embodied simulations of potential transformations of geometric objects while articulating their reasoning [1, 4]. This process deepens the connection between physical interaction and abstract mathematical thinking. Prior research on GeoGebra MR has primarily focused on examining the types of interactions that arise during system use, rather than conducting an in-depth analysis of the various forms of mathematical reasoning that emerge through these interactions [7, 8]. We extend that line of research by exploring how teachers (i.e., experts) and students (i.e., novices) compare in their interactions and mathematical reasoning within XR environments.

1.2 Differences in Categorizing and Representing Physics Problems: Experts vs. Novices

Chi and colleagues' study [9] highlights key differences in how novices and experts categorize and represent problems. Regarding categorization, novices focus on *First-order*, surface-level features, emphasizing literal objects and observable characteristics. In contrast, experts attend to *Second-order* features, abstracting underlying principles and categorizing them based on the deep structural relations of the laws of physics. Despite the qualitative differences in how experts and novices categorize problems, both groups construct enriched internal representations beyond the literal problem statement [9].

McDermott and Larkin's framework [10] identified four stages of representation in problem-solving: *literal representation*, which involves recognizing keywords and objects in the problem statement; *naïve representation*, which includes the spatial relationships of objects, often visualized through sketches; *scientific representation*, which introduces idealized objects and physical concepts necessary for forming equations; and *algebraic representation*, which develops formal equations to solve the problem. Previous research has shown that novices tend to create naïve representations focused on visible, familiar entities, whereas experts construct scientific and algebraic representations based on abstract principles that are meaningful within the context of formal physics [9].

Chi and colleagues [9] and McDermott and Larkin [10] offer contrasting views on the relationship between categorization and representation in problem-solving—whether these processes follow a fixed sequence or interact dynamically. This study adopts Chi and colleagues' perspective, which emphasizes a nonlinear interaction between representation stages, unlike the sequential model proposed by McDermott and Larkin. In this view, problems are tentatively categorized after an initial analysis, and subsequent representations are shaped by prior knowledge linked to that category [9]. For example, focusing on first-order features may lead to more concrete representations (e.g., literal and naïve), whereas focusing on second-order features may activate deeper schemata, resulting in more abstract representations (e.g., scientific and algebraic). Although originally applied to physics problem-solving, both frameworks are relevant to geometric reasoning due to the shared cognitive demands of abstracting principles, interpreting spatial relationships, and constructing formal representations.

1.3 Formalisms First View vs. Progressive Formalization in Geometric Reasoning

Nathan observed that many science and math curricula implicitly or explicitly assume that learning begins with the mastery of discipline-specific formalisms and progresses toward grounded, contextual applications; for example, algebra textbooks teach formal symbol manipulation before story problem-solving [11]. Nathan labeled this common pattern the *Formalisms First* (FF) [11]. Nathan critiques the FF view approach for teaching novice students because it privileging formal scientific and symbolic knowledge over applied, context-based knowledge and for fostering a "formalism-only" mindset that marginalizes other forms of reasoning and representation [11]. Empirical evidence also challenges the FF perspective, demonstrating that formalisms are more effectively learned by novices following prior, concrete experiences [7, 12].

As a promising alternative, Nathan advocates for the *Progressive Formalization* (PF) method, which emphasizes early experiences with concrete entities to provide accessible entry points for novices and foster a grounded understanding of the meaning of symbolic formalisms [11]. This approach supports the development of meaningful formal representations for later generalization and transfer [13]. Empirical studies further validate the effectiveness of PF, particularly in mathematical and geometric reasoning [7]. In the context of geometric reasoning, PF can be effectively applied by allowing students to physically interact with and manipulate geometric objects before introducing formal concepts. This gradual transition from hands-on exploration to abstraction deepens understanding and also promotes the ability to generalize geometric principles across contexts. By

leveraging the strengths of concreteness for intuitive understanding and accessibility, alongside the strengths of formalisms for generalization and computational efficiency, PF fosters deeper learning, better transfer, and sustained engagement [11, 13].

The FF approach is effective for experts because it reflects how they process information, focusing on second-order features—abstract principles used to categorize problems and apply scientific reasoning [9, 14]. In contrast, students often misapply formalisms due to limited conceptual understanding, relying on first-order features like literal objects and observable details [9, 11]. This gap suggests that PF can help students build a grounded understanding of formal concepts by connecting them to familiar, hands-on experiences [6]. Introducing formal ideas gradually *after* concrete engagement supports the integration of abstract reasoning into problem-solving.

Building on this, the research question is: *How do teachers and students categorize and represent geometric conjectures within GeoGebra MR through their physical interactions and reasoning discourse?* By examining these dynamics through the lenses of FF and PF, this study aims to identify the challenges students encounter when engaging with PF in GeoGebra MR. Ultimately, the findings will provide insights into designing XR environments that support effective geometric reasoning and improve learning outcomes.

2 Methods

2.1 Participants and Procedure

Teacher data collection was conducted in a laboratory space at a large Midwest university during the summer of 2023. Teachers (N=6) were placed into three dyads and tasked to reason through geometric conjectures using GeoGebra MR. Each dyad was assigned two geometric conjectures selected from secondary mathematics textbooks and validated by mathematicians and mathematics educators (see Table 1) [15].

Student data collection was conducted separately and involved 15 high school students (N = 15) enrolled in a summer psychology course at the same university in 2024. This research took place *in situ* during a unit on "Embodied Cognition." Students were assigned to one of eight dyads, one of which included a teacher who had previously participated in the teacher data collection phase. Before the study, students were informally interviewed by the course instructor about their confidence in mathematics. Notably, 6 out of 15 students reported struggling with or lacking interest in math. To balance mathematical ability, the instructor evenly assigned students to dyads. Student dyads were also tasked with reasoning through one of the same conjectures provided to teachers (Conjecture 2), along with four additional conjectures (see Table 1), using GeoGebra MR.

All teacher and student dyads utilized the same device (HoloLens 2) and were guided by the same research group. Their participation was recorded from first-person (MR headsets) and third-person (camcorder) perspectives. All videos were securely stored and transcribed for analysis.

Table 1. Geometric conjectures provided to teacher and student dyads.

Dyad	Conjectures
Teacher Dyads	1) The sum of the lengths of two sides of a triangle is always greater than the length of the third side. [True]
All Dyads	2) A cylinder with radius r and height h can be unrolled into a rectangle with length h and width 2πr. [True]
Student Dyads	3) Tripling the radius of a cylinder increases its volume by the same amount as tripling the height. [False]
	4) The volume of a cone is one-third the volume of a cylinder with the same base and height. [True]
	5) In a pyramid, the number of vertices is always the same as the number of faces. [True]
	6) When a square pyramid is filled to 50% of its height, it is filled to less than 75% of its volume. [False]

2.2 Materials

GeoGebra MR is an XR-based system designed to facilitate geometric reasoning through interactive manipulation of 3D objects (see Fig. 1). Users engage with geometric concepts by performing some operations through *direct manual manipulation*, such as scaling, skewing, and repositioning objects, and through *Algebra View menu selections*, sliders, and button selection, such as measuring, unfolding, filling, and inscribing objects.

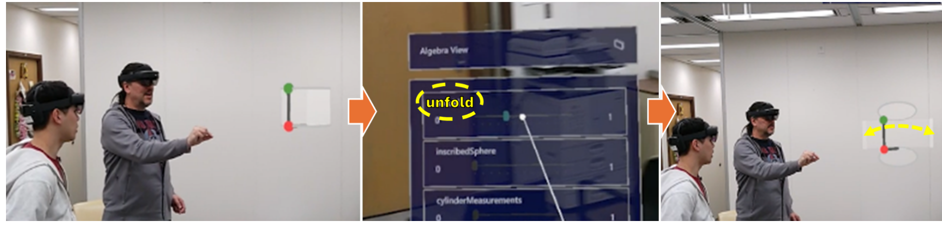

Fig. 1. Interactive features in Algebra View menu for reasoning about Conjecture 2 (Unfold).

2.3 Analysis

This study compares the reasoning processes of teacher and student dyads while engaging with *Conjecture 2*, which was explored by both groups. Additional conjectures were analyzed to support the primary findings. Participants' discourse and interactions with GeoGebra MR—including direct manual manipulations and Algebra View menu selections—were transcribed and systematically coded. The coding was guided by two frameworks: Chi and colleagues' framework [9] and McDermott and Larkin's Four Stages of Representation [10], revised to align with the context of this study (see Table 2). To assess ease of system interaction, the time each dyad took to complete their first manual manipulation and first Algebra menu operation was recorded.

Table 2. Codebook.

Framework	Code	Definition	Example
Categorization [9]	First-order	Surface-level features, emphasizing literal components of the given conjectures and observable characteristics of the virtual objects or features (e.g., Algebra View).	*We* UNFOLDED *it, and I see a* RECTANGLE. *I don't know what the* 2ΠR.
	Second-order	Correct underlying principles, patterns, equations, or concepts of the given conjectures.	*We can start from the* CIRCUMFERENCE *of the circle.*
Representation [10]	Literal	Simply reading the given conjectures or features of GeoGebra MR.	*Okay,* UNFOLD…
	Naïve	Understanding the spatial relationships of objects related to the given conjectures, visualized through gestures or interactions with virtual objects.	*They* LOOK THE SAME *when it's* UNFOLDED.
	Scientific	Identifying correct idealized objects and physical concepts related to the given conjectures.	2ΠR *is the* CIRCUMFERENCE OF THE CIRCLE *of cylinder*
	Algebraic	Applying correct formal equations to solve the given conjectures.	TWO TIMES 1.75 WOULD GIVE ME 3.5 TIMES Π. SO, 3.5 Π.

3 Results and Discussion

The results compare the reasoning processes of teacher and student dyads while solving a geometric problem: "*A cylinder with radius r and height h can be unrolled into a rectangle with length h and width 2πr.*" Specifically, the findings focus on differences in categorization and representation between student and teacher dyads.

3.1 Differences in Categorization between Student and Teacher Dyads

This section examines how teacher and student dyads categorize the geometric conjecture when first encountering the problem. First, we present the categorization process of teacher dyads, followed by that of student dyads, and then discuss the key differences between them.

Teacher Dyads. The excerpt below illustrates how the teacher dyad categorized a geometric conjecture.

Teacher 1: We need to see *the measurement of the cylinder* and *unwrap* it, right?
Teacher 2: Can we do *circumference measurement*?
Teacher 1: We would need to see two things, *the dimensions of this rectangle, the length and the width*.

Teacher 2: Oh, it has a radius here. Then we would do 1.75 as the radius then *circumference would be 2πr*, so two times 1.75 would give me 3.5 times π. So, 3.5 π. And we need to show that that is equal to the length of this rectangle. Unfortunately, we don't have the length of this rectangle, right?

Teacher 1: Well, we have this *surface area* (of the cylinder), so we can work this all backwards. Because we know *the area of the circles*, we can subtract by *the area of the rectangle*. And then we know the height so that we can divide.

Teacher 1 suggested identifying the measurements of the cylinder's components and unwrapping the cylinder accordingly. This approach incorporates both first-order features (i.e., unwrapping, which directly relates to the physical components of the conjecture) and second-order features (i.e., measurement concepts such as circumference and length, which involve abstract mathematical principles). Teacher 2 elaborated Teacher 1's suggestion by specifying the circumference measurement, demonstrating a focus on second-order features. Building on this, Teacher 1 further considered additional components (the dimensions of the rectangle) necessary to construct the equation. Based on this discussion, Teacher 2 completed the equation as a second-order feature, multiplying the radius (1.75) by 2 and π, arriving at 3.5π. This led the reasoning process toward finding evidence to establish the equivalence between the circumference and the rectangle's length. Teacher 1 then extended this reasoning by introducing another equation for surface area measurement: *Surface area of the cylinder - 2 × the area of the circles = the area of the rectangle*. This reasoning process highlights that the teacher dyad's quickly and easily identified second-order features relevant to the given geometric conjecture, while still acknowledged both first- and second-order elements in their reasoning.

Student Dyads. The excerpt below illustrates how the student dyad categorized a geometric conjecture.

Student 1: You could see the box, right? The *rectangle* … they look the exact same.

Student 2: They look the same when it's *unfolded*… But *π* and *radius* wouldn't be used to find the width of *rectangle*, it's only used for a circle.

Student 1 began reasoning by focusing on the visualized feature (a first-order feature) of the virtual object, noting that the width of the rectangle appeared to be the same as the circumference of the cylinder's bases. Student 2 focused on the components of the conjecture (first-order features: π, radius, and rectangle), leading the reasoning to incorrect logic (i.e., the width of the rectangle cannot be determined using π and the radius). This discourse shows that the student dyad focused solely on the first-order features of the virtual object or the conjecture without recognizing the underlying principle (second-order features), which contrasts with the teachers' reasoning.

These findings suggest that teachers and students categorize the geometric conjecture differently when first encountering the problem. Specifically, teachers recognized that the conjecture pertained to the *circumference of a circle* and, by extension, the *surface area of a cylinder*. In contrast, students' categorization was predominantly guided by surface-level features, particularly the terms *π* and *radius*. A similar pattern emerged with Conjecture 4. Half of the student dyads immediately focused on the first-order feature "the same base and height" but failed to identify any second-order features. For example, one student remarked, "So it has t*he same base and height*, I feel like that's always true," while another stated, "I think it's true because they said *the same base and height.* If they just said cylinder, it wouldn't be right." The frequency analysis further supports this tendency. The ratio of first-order to second-order feature usage for Conjecture 2 was 0.32 for teachers and 0.82 for students, indicating that teachers relied significantly more on second-order reasoning compared to students. These results illustrate that teachers categorize the conjecture based on its underlying conceptual structure, whereas students' categorization is primarily based on observable elements of the virtual objects or conjecture.

3.2 Differences in Constructing Representation between Student and Teacher Dyads

This section examines how teacher and student dyads construct representations related to the given geometric conjecture. Specifically, the focus is on how they establish connections between naïve representations (concrete physical interactions) and scientific or algebraic representations (abstract concepts). Furthermore, the discussion explores how student dyads differ in making these connections and provides design suggestions for enhancing XR-based geometry learning environments.

Teacher and student dyads demonstrated distinct approaches in connecting scientific or algebraic representations with naive representations in GeoGebra MR. Teachers effectively connected geometric concepts (e.g., the equation for the circumference of a circle) to the dynamic states and conditions of geometric object representations (e.g., a rectangle folding around the circumference of a cylinder's bases). They were more likely to construct scientific and algebraic representations directly, guided primarily by their initial categorization focused on second-order features. After forming these abstract representations, they reversely explored how to translate these concepts into naïve representations by utilizing GeoGebra MR's interactive features. For example,

teachers applied abstract concepts and equations (e.g., circumference of a circle = $2\pi r$) to interpret and manipulate the spatial relationships of geometric objects (e.g., unwrapping the cylinder into a rectangle between two tangent circles). This dynamic process demonstrates how GeoGebra MR enabled teachers to ground their advanced reasoning in interactive, tangible experiences, refining their understanding through a reciprocal relationship between abstract and concrete representations. GeoGebra MR facilitated these interactions by enabling users to visualize and manipulate spatial relationships (e.g., unfolding) in geometric objects, thereby supporting the development of naïve representations informed by scientific reasoning. Additionally, through its virtual dashboard (Algebra View), GeoGebra MR allowed users to perform algebraic operations and measure geometric properties, embedding abstract concepts (e.g., measurements) into tangible 3D objects. This capability fostered connections between scientific, algebraic, and naïve representations. Teachers highly valued these interactive features, incorporating them smoothly into their reasoning processes to bridge physical interactions with idealized geometric concepts and equations. For example, one teacher immediately connected the naïve representation (the unwrapped cylinder) to a geometric principle (the width of the rectangle is always equal to the circumference of the base) even before researchers introduced the conjecture, stating, "Are you opening that (cylinder)? It's so useful because it shows the (transformation into) a rectangle." Similarly, another teacher quickly linked the Algebra View menu function (cylinder measurements) to a specific equation for proving the rectangle's width, explaining, "We have this surface area (measurement function), so we can work this (the equation) all backwards because we know the area of the circle…"

In contrast, the majority of students showed little interest in connecting naïve representations immediately observable with GeoGebra MR to scientific and algebraic representations, or vice versa. Seven students appeared to overlook the *transition from naïve representations to scientific and algebraic representations*. Their focus remained primarily on the literal representations of the stated conjecture, driven by their initial surface-level categorization. For example, one student remarked, "So, we *unfolded* it, and I see a *rectangle*. I don't know what the $2\pi r$. I think it's true because there are *two* circles on the cylinder, and each circle could be a *radius*?" This student referenced the literal components of the conjecture (e.g., unfold, rectangle, $2\pi r$) and visualized the naïve representation by unfolding the cylinder. However, they failed to connect the literal component $2\pi r$ to the idealized equation for the circumference of a circle. Their reasoning remained fixated on the observable characteristics of the naïve representation ("two" circles), leading to an incorrect interpretation ("two" πr). This suggests that the student focused on surface-level features (literal and naïve representations) to reason through the conjecture, showing little interest in elaborating their reasoning with grounding principles from scientific and algebraic representations. Similarly, another student stated, "$2\pi r$ sounds right because that is one of the things they use for finding the word I can't remember about the circle, and it looks like it would be too big to be two. So, I feel like putting in the π would be correct." While this student recognized the literal component ($2\pi r$), they quickly dismissed the need to engage further with the relevant geometric concept (the circumference of a circle) demonstrating a lack of interest in reasoning beyond the surface level.

On the other hand, four students appeared to overlook the *transition from scientific and algebraic representations to naïve representations*. These students attempted to retrieve relevant formal concepts from prior math education based solely on the literal terms of the conjecture, influenced by their initial surface-level categorization. However, most of them developed misconceptions, superficially referencing literal terms ($2\pi r$) without grounding their reasoning in the conjecture's geometric principles. For instance, one student confused the circumference of the circle with its diameter, explaining, "The diameter is 2r, but then you still have π, so you subtract, there is an extra π." Similarly, another student incorrectly stated, "π and radius wouldn't be used to find the width of a rectangle." These misconceptions led students to become indifferent toward, or even distrust, the naïve representations they visualized and manipulated in GeoGebra MR and instead adhere to their preexisting incorrect assumptions. The student exemplified this, stating, "They look the same when it's unfolded. But…" These students demonstrated confidence in their mathematical abilities, which may reflect a strong reliance on a *formalisms-only mindset* in geometric reasoning [11]. Despite GeoGebra MR providing concrete sensorimotor experiences to represent abstract concepts in the conjecture, these students often defaulted to their formalized understanding and showed little interest in situating idealized interpretations within the concrete context of the problem. These two findings highlight students' limited interest in connecting naïve representations to scientific and algebraic representations, underscoring the need for both instructional and technological strategies that effectively scaffold this transition.

Only two student dyads successfully connected literal and naïve representations to scientific and algebraic representations. A distinguishing feature of these dyads was their stronger sense of purpose in constructing naïve representations using GeoGebra MR. For instance, in Conjecture 4, most other students did not engage in discussions about constructing naïve representations, instead relying solely on the literal representation ("the same base and height"). This difficulty likely stemmed from the absence of explicit operational cues in the conjecture's literal representation "The volume of a cone is one-third the volume of a cylinder with the same base and height"

[9]. In contrast, these dyads actively leveraged the operational functions in GeoGebra MR to construct naïve representations. Their discussions reflected an exploratory approach, with comments such as, "What if we change the number of sides?" and "What about the volume area measurement?" This success may be attributed to their relatively greater ease in navigating and manipulating the GeoGebra MR interface. On average, these two dyads took approximately 8 seconds to perform their first operation, compared to an average of 19 seconds for the other groups, with three participants failing to operate the system entirely. These observations highlight the importance of incorporating explicit operational cues (e.g., unfold, fill) into literal representation to assist students in developing naïve representations. A well-designed geometric learning environment should offer accessible entry points for constructing naïve representations through concrete sensorimotor engagement with geometric spatial relationships [11]. Such experiences serve as grounded symbolization, enabling the development of meaningful scientific and algebraic representations that support later generalization and transfer [11, 13].

4 Conclusion

This study highlights two key differences in the reasoning processes of teachers and students when using GeoGebra MR. First, teachers identified both first-order and second-order features during categorization, while most students focused only on first-order features. In the reasoning processes for Conjecture 2, teachers demonstrated a lower first-order to second-order feature usage ratio (0.32) compared to students (0.82), indicating teachers relied more on underlying principles, whereas students focused on surface-level details. Second, teachers successfully constructed and connected scientific and algebraic representations to naïve representations by manipulating spatial relationships and algebraic functions. In contrast, most students failed to integrate these representations, either stopping at naïve reasoning or misapplying formal concepts without grounding them in visual representations.

While this study is limited in its generalizability due to the small number of teachers and students examined within a constrained geometry task, understanding these differences offers valuable insights for both learning scientists and educational practitioners. For researchers, the findings expand the literature on how XR environments can support learners' cognitive development, especially in bridging concrete and abstract reasoning. For practitioners, the study highlights the need to scaffold progressive formalization (PF)—the gradual shift from hands-on exploration to formal reasoning. Since students often rely on first-order features, designing conjectures with explicit operational cues (e.g., "unfold" in Conjecture 2) can provide accessible entry points for PF. Additionally, visually integrating scientific representations with naïve representations—such as overlaying $2\pi r$ on a circle—can help students connect geometric concepts to formal equations. The success of two dyads, who effectively connected naïve and formal reasoning, underscores the value of providing manipulation tutorials and technological support to enhance student interaction with XR tools. Addressing these instructional and design needs can help students develop a more grounded understanding of formal mathematical reasoning. These insights can guide the creation of XR-based educational tools that improve learning outcomes in STEM education.

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